Exam Lie Groups in Physics

Date	November 4, 2019
Room	BB 5161.0165
Time	15:00 - 18:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	6	2a)	6	3a)	8	4a)	8
1b)	6	2b)	8	3b)	6	4b)	8
1c)	6	2c)	8			4c)	8
1d)	6	2d)	6				

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Problem 1

(a) Consider the sets of real numbers R and positive real numbers R^+. Indicate the composition laws under which these sets form Lie groups. Explain your answers and explain why division cannot be a composition law.

(b) Show that $\mathsf{R} \cong \mathsf{R}^+$.

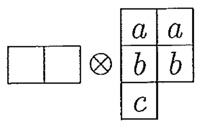
(c) Show that $\mathsf{R}/\mathsf{Z} \cong U(1)$, where Z denotes the group of integers and $U(1)$ the group of unitary 1×1 matrices.

(d) Consider the cosets of $U(1)$ in the group $G = \mathsf{C}\backslash\{0\}$ of nonzero complex numbers. Show which group is formed by $G/U(1)$.

Problem 2

Consider the Lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

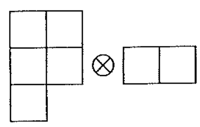
(a) Consider the following direct product of irreps of the Lie algebra $su(n)$:



Write down all allowed sequences ("words") consisting of the letters a, a, b, b, c.

(b) Decompose the above direct product of irreps into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(c) Do the same direct product but now in reverse order:



(d) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(3)$ and $su(4)$. Indicate the complex conjugate and inequivalent irreps whenever appropriate.

Problem 3

Consider the Cartan matrix $A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$, where the α_i denote the simple roots, for the complex Lie algebra $\widetilde{sp}(2)$:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

(a) Use this matrix to obtain the root diagram of $\widetilde{sp}(2)$ by using Weyl reflections.

(b) Deduce the dimension of the Lie algebra $\widetilde{sp}(2)$, of its compact real form $sp(2)$ and of the corresponding Lie group.

Problem 4

Consider the group of Lorentz transformations $L^{\mu}{}_{\nu}$.

(a) Demonstrate that invariance of the Minkowski metric under Lorentz transformations implies that $(L^0{}_0)^2 \geq 1$.

Consider the Lorentz algebra given by

$$[J^j, J^k] = i\epsilon_{jkl} J^l, \quad [J^j, K^k] = i\epsilon_{jkl} K^l, \quad [K^j, K^k] = -i\epsilon_{jkl} J^l.$$

(b) Show by explicit calculation that the following two operators are Casimir operators of the Lorentz group, i.e. commute with all the generators of the Lorentz algebra:

$$C_1 = \vec{J}^2 - \vec{K}^2, \quad C_2 = 2\vec{J} \cdot \vec{K}$$

(c) Explain why the Casimir operators can be used to label the irreducible representations.